A static memory sparse spectral method for time-fractional PDEs

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Talk structure

1. Motivation from photoacoustic ultrasound imaging
2. The Yuan-Agrawal method for Caputo derivatives
3. A static memory, sparse and recursive solver
4. Numerical experiments
1. Motivation from photoacoustic ultrasound imaging
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![Diagram of photoacoustic ultrasound imaging process]

- **Pulsed laser excitation**
- **Ultrasonic emission**
- **Ultrasonic detection**

**Steps:**
1. Laser/RF pulse
2. Absorption
3. Thermal expansion
4. Acoustic waves
5. Ultrasonic detection
6. Image formation
1. Motivation from photoacoustic ultrasound imaging

Absorption of compressional and shear waves in viscoelastic media tend to follow a frequency power law\[^1\].

\[
\mathbf{u} = \nabla \phi + \nabla \times \Psi
\]

\[
0 = \nabla \left( \frac{\partial^2 \phi}{\partial t^2} - c_p^2 \nabla^2 \phi - \tau_p c_p^2 \frac{\partial^{y-1}}{\partial t^{y-1}} \nabla^2 \phi \right) + \nabla \times \left( \frac{\partial^2 \Psi}{\partial t^2} - c_s^2 \nabla^2 \Psi - \tau_s c_s^2 \frac{\partial^{y-1}}{\partial t^{y-1}} \nabla^2 \Psi \right).
\]
1. Motivation from photoacoustic ultrasound imaging

Image Credit: k-wave MATLAB package documentation
http://www.k-wave.org/documentation/example_ivp_loading_external_image.php
1. Motivation from photoacoustic ultrasound imaging

Image Credit: k-wave MATLAB package documentation
http://www.k-wave.org/documentation/example_pr_2D_tr_circular_sensor.php
2. The Yuan-Agrawal method for Caputo derivatives
Definition (Caputo derivative):

\[
\frac{\partial^\alpha}{\partial t^\alpha} f(t) = D_C^\alpha f(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^t (t - s)^{([\alpha] - \alpha - 1)} f([\alpha]) (s) \, ds,
\]

Inherent challenge: Non-local in time $\rightarrow$ Memory accumulation
Treeby & Cox’s solution:
Transform the \textit{time}-fractional into a \textit{space}-fractional problem.

\[
\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \phi - \nabla^2 \phi - \tau \frac{\partial^{y-1}}{\partial t^{y-1}} \nabla^2 \phi = 0. 
\]

\[
\downarrow \\

\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi - \tau_1 (-\nabla^2)^{y/2} \frac{\partial}{\partial t} \phi - \tau_2 (-\nabla^2)^{(y+1)/2} \phi = 0.
\]
Treeby & Cox’s solution:
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\]

\[
\mathcal{F}_{x,t}\left\{ \frac{\partial^y g(x,t)}{\partial t^y} \right\} = (-i\omega)^y G(k, \omega).
\]

\[
\omega \tau \ll 1
\]

\[
\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \phi - \nabla^2 \phi - \tau_1 (-\nabla^2)^{y/2} \frac{\partial}{\partial t} \phi - \tau_2 (-\nabla^2)^{(y+1)/2} \phi = 0.
\]
2. The Yuan-Agrawal method for Caputo derivatives

THEOREM 3.1 (Generalized Yuan–Agrawal–Caputo derivative). Let \( \alpha > 0, \alpha \notin \mathbb{N} \) and \( f \in C^{[\alpha]}[0, T] \). Then the Caputo fractional derivative of \( f \) can be expressed as

\[
\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t) = \int_0^\infty \phi(w, t) dw,
\]

where the function \( \phi_f : (0, \infty) \times [0, T] \rightarrow \mathbb{R} \) is defined by

\[
\phi_f(w, t) := \frac{(-1)^{[\alpha]} 2 \sin(\pi \alpha)}{\pi} w^{2\alpha - 2[\alpha] + 1} \int_0^t e^{-w^2(t-\tau)} \frac{\partial^{[\alpha]}}{\partial \tau^{[\alpha]}} f(\tau) d\tau.
\]

Furthermore, for fixed \( w > 0 \) the function \( \phi_f(w, t) \) satisfies the (non-fractional) differential equation

\[
(3.1) \quad \frac{\partial}{\partial t} \phi_f(w, t) = -w^2 \phi_f(w, t) + \frac{(-1)^{[\alpha]} 2 \sin(\pi \alpha)}{\pi} w^{2\alpha - 2[\alpha] + 1} \frac{\partial^{[\alpha]}}{\partial t^{[\alpha]}} f(t),
\]

with initial condition \( \phi(w, 0) = 0 \).

2. The Yuan-Agrawal method for Caputo derivatives

Criticism of the Yuan-Agrawal method:
Gauss-Laguerre quadrature tends to work very poorly for these problems and requires prohibitively large number of nodes.

A solution to this was proposed by Diethelm\textsuperscript{[1]} (and Birk & Song\textsuperscript{[2]}):

\[
\int_0^\infty \phi_f(w, t)\,dw = \int_{-1}^1 (1 - \kappa)^{\bar{\alpha}}(1 + \kappa)^{-\bar{\alpha}} \phi_f(\kappa, t)\,d\kappa,
\]

\[
\bar{\phi}_f(\kappa, t) := 2(1 - \kappa)^{-\bar{\alpha}}(1 + \kappa)^{\bar{\alpha}-2}\phi_f\left(\frac{1 - \kappa}{1 + \kappa}, t\right).
\]

\textsuperscript{[1]} Diethelm 2008, Numerical Algorithms
\textsuperscript{[2]} Birk & Song 2010, Comput. Mech
3. A static memory, sparse and recursive solver
Restate Yuan-Agrawal method in recursive form:

\[ \frac{\partial^\alpha}{\partial t^\alpha} f(t) \approx \sum_{j=1}^{L} A_j \int_{0}^{t} e^{-s_j^2(t-\tau)} \frac{\partial^{[\alpha]}}{\partial \tau^{[\alpha]}} f(\tau) d\tau = \sum_{j=1}^{L} A_j \psi_j(t), \]

\[ \psi_j(t) := \int_{0}^{t} e^{-s_j^2(t-\tau)} \frac{\partial^{[\alpha]}}{\partial \tau^{[\alpha]}} f(\tau) d\tau. \]

\[ \psi_j(t) = e^{-s_j^2 \Delta t} \psi_j(t - \Delta t) + \int_{t-\Delta t}^{t} e^{-s_j^2(t-\tau)} \frac{\partial^{[\alpha]}}{\partial \tau^{[\alpha]}} f(\tau) d\tau. \]
Resolve spatial dependence in multivariate orthogonal polynomials:

\[ f(n\Delta t, x) = P(x) f(n\Delta t), \]

\[ \psi_j(n\Delta t, x) = P(x) \psi_j(n\Delta t). \]

We obtain:

\[
\left( \frac{\partial^\alpha f}{\partial t^\alpha} \right)(n\Delta t, x) \approx P(x) \sum_{j=1}^{L} A_j \left( e^{-s_j^2 \Delta t} \psi_j^{n-1} + \frac{1-e^{-s_j^2 \Delta t}}{s_j^2 \Delta t} (f^n - f^{n-1}) \right),
\]

\[ \psi(n\Delta t, x) = P(x) \psi(n\Delta t) \approx P(x) \psi_j^n = P(x) \left( e^{-s_j^2 \Delta t} \psi_j^{n-1} + \frac{1-e^{-s_j^2 \Delta t}}{s_j^2 \Delta t} f^n - f^{n-1} \right). \]
The memory requirements of computing a Caputo derivative:

\[ L(2+K)+2K, \]

where

- \( L \) ... number of quadrature pts.
- \( K \) ... degree of polynomial approx.
4. Numerical experiments
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Discrete particle dynamics described by Newtonian dynamics:

(a) $f(t) = t^2$, $L = 65$, $\alpha = \frac{2}{3}$

(b) $f(t) = e^t$, $L = 60$, $\alpha = \frac{1}{2}$
\[ \left( \frac{\partial^{\alpha} f}{\partial t^{\alpha}} \right)(T, x) - P(x) P_K \sum_{j=1}^{L} A_j \psi_j^N \right| \leq \frac{M(x)}{(2L)!} + C(x) LT \Delta t + err_{P,K} \left( \sum_{i=1}^{L} A_j \psi_j (T, x) \right). \]

4. Numerical experiments

(a) \( f(t) = t^2 \), \( \alpha = \frac{1}{4} \)

(b) \( f(t) = e^t \), \( \alpha = \frac{1}{2} \)
Discretize our equation of interest on the unit disk:

\[
\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} f - \Delta f + \tau \frac{\partial^{\alpha}}{\partial t^\alpha} f = 0,
\]

\[
Z^{(1)}(r, \theta) \left( \left( \frac{1}{c_0^2 \Delta t} + \tau \left( \sum_{j=1}^{L} A_j \frac{1-e^{-s_j^2 \Delta t}}{s_j^2} \right) \right) C - (\Delta t) \Delta \right) f^n =
\]

\[
Z^{(1)}(r, \theta) C \left( \left( \frac{2}{c_0^2 \Delta t} + \tau \sum_{j=1}^{L} A_j \frac{1-e^{-s_j^2 \Delta t}}{s_j^2} \right) f^{n-1} - \frac{f^{n-2}}{c_0^2 \Delta t} - \tau \sum_{j=1}^{L} A_j e^{-s_j^2 \Delta t} \psi_j^{n-1} \right),
\]
Modeling circular sensor arrays:

(a)

(b)
Next goals:
- Fully realized image reconstruction
- Comparison with fractional Laplacian methods
- Combine with spectral element methods on annular domains
A static memory sparse spectral method for time-fractional PDEs

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